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Oblique instability of periodic waves in shallow water

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Periodic permanent waves in shallow water are stable to periodic disturbances in the same direction, but are unstable to certain oblique periodic disturbances. A computer-assisted stability analysis is made of such waves for oblique disturbances with wavelengths comparable to and long compared with the fundamental wavelength. Regions of instability are calculated, and an explanation is given for the occurrence of instability. It is shown that disturbances in the same direction with a small margin of stability may cause a greater modification to the permanent wave in practice than do oblique unstable disturbances.

1. Introduction

Stokes waves of fundamental wavelength $2\pi l$ on water of uniform depth h are stable to long disturbances in the same direction provided that $h/l < 1{\cdot}36$ (Whitham 1974, § 16.11), but are unstable to certain oblique long disturbances for almost all values of h/l (Benney & Roskes 1969; Hayes 1973; Davey & Stewartson 1974). Long Stokes waves ($h/l \ll 1$) have only an infinitesimal amplitude a, since $a/h \ll (h/l)^2$ is a necessary condition for the existence of such waves (Whitham 1974, § 13.13). In an earlier paper (Bryant 1974, referred to as I), the range of long permanent waves of small but finite amplitude was calculated, and their stability to comparable and long disturbances in the same direction was investigated. It was shown that the waves are stable to such disturbances, but that the margin of stability decreases towards zero as the length of the fundamental wave or the length of the disturbance becomes large compared with the depth.

The method of calculation in I was a classical linear stability analysis, but because of the complexity of the coefficients and the large number of variables, a computer was used to assist the calculations. The same method is extended here to include oblique disturbances applied to the above range of permanent waves. The three investigations cited above were concerned only with long disturbances applied to Stokes waves, the disturbances being long in the sense of having wavenumbers small compared with the fundamental wavenumber of the Stokes wave. The present method of calculation extends their analysis to disturbances of wavenumber small compared with or comparable to the fundamental wavenumber, and to long permanent waves with finite but small amplitudes greater than those permissible for Stokes waves.

Phillips (1974) explained the mechanism for instability of Stokes waves in terms of resonant interactions between the disturbance and the first two harmonics of the Stokes wave. If the first two harmonics have wavenumbers $\mathbf{k}$ and $2\mathbf{k}$, the first approximations to their frequencies are $\omega(\mathbf{k})$ and $2\omega(\mathbf{k})$, where $\omega(\mathbf{k})$ is the frequency given by the linear dispersion relation. For a disturbance of wavenumber $\boldsymbol{\kappa}$, the condition for resonance is

$$\omega(\mathbf{k}-\boldsymbol{\kappa})+\omega(\mathbf{k}+\boldsymbol{\kappa}) = 2\omega(\mathbf{k}). \tag{1.1}$$

A mismatch with this linear resonance condition occurs because of the dependence of all three frequencies on amplitude as a result of quadratic interactions. Instability occurs, therefore, in the neighbourhood of the curve in wavenumber space described by (1.1) (Phillips 1974, figure VII.13).

2. Linear stability analysis

Periodic waves of fundamental wavelength $2\pi l$ are generated in water of mean depth h bounded above by a free surface and below by a smooth horizontal bed. The two principal non-dimensional ratios are $\epsilon = a/h$ and $\mu = h/l$, where a is a measure of wave amplitude. The horizontal non-dimensional co-ordinates x_1 and x_2 in the mean free surface are measured in units of l, and time t is measured in units of l/c_0, where $c_0 = (gh)^{\frac{1}{2}}$ is the linear long-wave velocity. The governing equations are of the same form as those in I [equations (2.1)], extended to include one more horizontal co-ordinate. The periodic surface displacement is expanded in the Fourier series

$$\eta(\mathbf{x},t) = \tfrac{1}{2}\Sigma A(\mathbf{k},t)\exp\{i\mathbf{k}.(\mathbf{x}-\mathbf{c}t)\}+*, \tag{2.1}$$

where $\mathbf{k} = (k_1, k_2)$ is in units of $1/l$, $\mathbf{x} = (x_1, x_2)$, $\mathbf{c}$ is to be identified with the velocity of permanent waves in units of c_0 and the asterisk denotes the complex conjugate. On substitution into the governing equations, the Fourier amplitudes are found to satisfy (with $D = d/dt$)

$$\begin{aligned} DA(\mathbf{k})-i(\mathbf{k}.\mathbf{c}-\omega(\mathbf{k}))A(\mathbf{k}) = &-\tfrac{1}{2}i\epsilon\,\Sigma R(\mathbf{k},-\mathbf{l})A(\mathbf{l})A(\mathbf{k}-\mathbf{l}) \\ &-i\epsilon\,\Sigma R(\mathbf{k},\mathbf{l})A^*(\mathbf{l})A(\mathbf{k}+\mathbf{l})+O(\epsilon^2), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} R(\mathbf{k},\mathbf{l}) = &\frac{[\omega(\mathbf{k}+\mathbf{l})-\omega(\mathbf{l})][\mathbf{k}.\mathbf{l}\,\omega(\mathbf{k}+\mathbf{l})-\mathbf{k}.(\mathbf{k}+\mathbf{l})\,\omega(\mathbf{l})]+\omega^2(\mathbf{k})\,\mathbf{l}.(\mathbf{k}+\mathbf{l})}{2\omega(\mathbf{l})\,\omega(\mathbf{k}+\mathbf{l})\,[\omega(\mathbf{k})+\omega(\mathbf{k}+\mathbf{l})-\omega(\mathbf{l})]} \\ &-\mu^2\frac{\omega^2(\mathbf{k})\,[\omega^2(\mathbf{l})-\omega(\mathbf{l})\,\omega(\mathbf{k}+\mathbf{l})+\omega^2(\mathbf{k}+\mathbf{l})]}{2[\omega(\mathbf{k})+\omega(\mathbf{k}+\mathbf{l})-\omega(\mathbf{l})]}, \end{aligned} \tag{2.3}$$

$\omega(\mathbf{k}) = \{(k/\mu)\tanh k\mu\}^{\frac{1}{2}}$, $k = (k_1^2+k_2^2)^{\frac{1}{2}}$, and $\omega(-\mathbf{l})$ is to be interpreted as $-\omega(\mathbf{l})$ in calculating $R(\mathbf{k},-\mathbf{l})$.

The Fourier series for the permanent wave is

$$\eta(x,t) = \frac{1}{2}\sum_{k=1}^{\infty} a_k \exp\{ik(x_1-ct)\}+*, \tag{2.4}$$

where c and the amplitudes a_k are calculated for the given ϵ and μ as in I (§ 3). When the permanent wave is perturbed by a periodic disturbance of wavenumber $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$, the perturbed surface displacement is written as

$$\begin{aligned} \eta(\mathbf{x},t) = &\frac{1}{2}\sum_{k=1}^{\infty}\hat{A}(k\mathbf{l}_1-\boldsymbol{\kappa},t)\exp\{i(k\mathbf{l}_1-\boldsymbol{\kappa}).(\mathbf{x}-c\mathbf{l}_1t)\} \\ &+\frac{1}{2}\sum_{k=1}^{\infty}a_k\exp\{ik(x_1-ct)\} \\ &+\frac{1}{2}\sum_{k=0}^{\infty}\hat{A}(k\mathbf{l}_1+\boldsymbol{\kappa},t)\exp\{i(k\mathbf{l}_1+\boldsymbol{\kappa}).(\mathbf{x}-c\mathbf{l}_1t)\}+*, \end{aligned} \tag{2.5}$$

where $\mathbf{i}_1$ is a unit vector in the x_1 direction. This representation has been chosen so that the coefficients in (2.6) are constants. Although it is the dependent variable $\hat{A}(k\mathbf{i}_1 \pm \boldsymbol{\kappa}, t)\exp\{-i[(k\pm\kappa_1)c-\omega(k\mathbf{i}_1\pm\boldsymbol{\kappa})]t\}$ which varies on the slow time scale ϵt, $\hat{A}$ itself is written as being dependent on ϵt in the examples following. This simplification is possible because $c = 1+O(\epsilon)$, $\omega(\mathbf{k}) = k(1+O(k\mu)^2)$ and hence the exponent is $O(\epsilon)$ whenever μ^2 is $O(\epsilon)$ and k is not too large. Substitution of the perturbed harmonics in (2.2), followed by linearization in $\hat{A}$, yields

$$\begin{aligned} D\hat{A}(k\mathbf{i}_1-\boldsymbol{\kappa},t)-i[(k-\kappa_1)c-\omega(k\mathbf{i}_1-\boldsymbol{\kappa})]\,\hat{A}(k\mathbf{i}_1-\boldsymbol{\kappa},t) \\ = -i\epsilon\sum_{l=1}^{k-1} R(k\mathbf{i}_1-\boldsymbol{\kappa},\,-l\mathbf{i}_1)\,a_l\hat{A}(\{k-l\}\,\mathbf{i}_1-\boldsymbol{\kappa}) \\ -i\epsilon\sum_{l=1}^{\infty} R(k\mathbf{i}_1-\boldsymbol{\kappa},l\mathbf{i}_1)\,a_l\hat{A}(\{k+l\}\,\mathbf{i}_1-\boldsymbol{\kappa}) \\ -i\epsilon\sum_{l=0}^{\infty} R(k\mathbf{i}_1-\boldsymbol{\kappa},l\mathbf{i}_1+\boldsymbol{\kappa})\,a_{k+l}\hat{A}^*(l\mathbf{i}_1+\boldsymbol{\kappa})+O(\epsilon^2), \quad k = 1,2,\ldots, \end{aligned} \tag{2.6a}$$

$$\begin{aligned} D\hat{A}(k\mathbf{i}_1+\boldsymbol{\kappa},t)-i[(k+\kappa_1)c-\omega(k\mathbf{i}_1+\boldsymbol{\kappa})]\,\hat{A}(k\mathbf{i}_1+\boldsymbol{\kappa},t) \\ = -i\epsilon\sum_{l=1}^{\infty} R(k\mathbf{i}_1+\boldsymbol{\kappa},\,-l\mathbf{i}_1)\,a_l\hat{A}(\{k-l\}\,\mathbf{i}_1+\boldsymbol{\kappa}) \\ -i\epsilon\sum_{l=1}^{\infty} R(k\mathbf{i}_1+\boldsymbol{\kappa},l\mathbf{i}_1)\,a_l\hat{A}(\{k+l\}\,\mathbf{i}_1+\boldsymbol{\kappa}) \\ -i\epsilon\sum_{l=1}^{\infty} R(k\mathbf{i}_1+\boldsymbol{\kappa},l\mathbf{i}_1-\boldsymbol{\kappa})\,a_{k+l}A^*(l\mathbf{i}_1-\boldsymbol{\kappa})+O(\epsilon^2), \quad k = 0,1,2,\ldots. \end{aligned} \tag{2.6b}$$

Equations ($2.6a^*, b$) are a set of first-order linear differential equations for the perturbation harmonics $\hat{A}(\boldsymbol{\kappa})$, $\hat{A}^*(\mathbf{i}_1-\boldsymbol{\kappa})$, $\hat{A}(\mathbf{i}_1+\boldsymbol{\kappa}), \ldots, \hat{A}^*(k\mathbf{i}_1-\boldsymbol{\kappa})$, $\hat{A}(k\mathbf{i}_1+\boldsymbol{\kappa}), \ldots$.

The $2n+1$ equations for the first $2n+1$ perturbation harmonics may be solved by the normal-mode method (I, § 4) by seeking solutions with time dependence $\exp(i\lambda\epsilon t)$. The number of equations was chosen to be sufficiently large that solutions are reproduced within the desired accuracy (usually 10^{-4}) when n is increased. Instability occurs when a pair of eigenvalues λ is complex.

3. Examples

The regions of instability are now calculated for two permanent waves of the same amplitude ($\epsilon = 0{\cdot}05$) but of differing wavelengths ($\mu = 0{\cdot}5$, $0{\cdot}25$). The first three harmonics of the permanent wave for which $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}5$ are 0·98, 0·17 and 0·02. This value of μ is sufficiently large that the harmonics approximate the theoretical values for a Stokes wave (Whitham 1974, § 13.13). The first five harmonics of the permanent wave for which $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}25$ are 0·84, 0·39, 0·14, 0·05 and 0·02. These harmonics differ from those for a Stokes wave since the condition $\epsilon \ll \mu^2$ is no longer valid here.

The region of instability for the first permanent wave is sketched in figure 1. The outer fine curve is the outer bound of instability and the inner fine curve is the contour on which $\lambda_i = 2\times10^{-3}$ for the unstable mode whose time dependence is

$$\exp\{i(\lambda_r - i\lambda_i)\,\epsilon t\}.$$

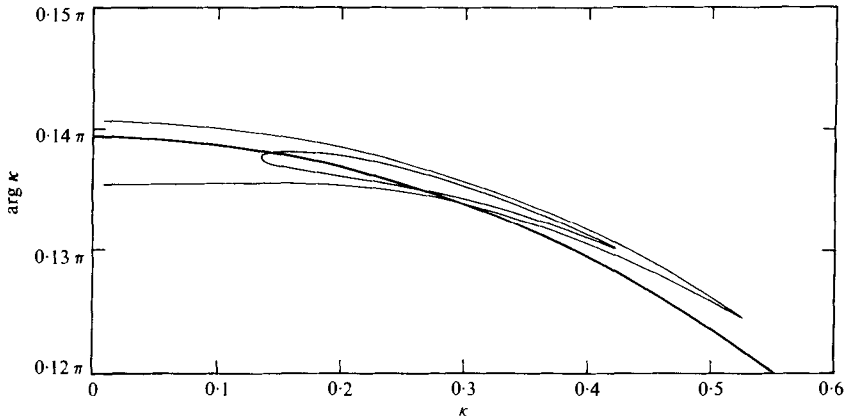

FIGURE 1. The region of instability for the permanent wave with $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}5$. The bold curve is the curve of linear resonance.

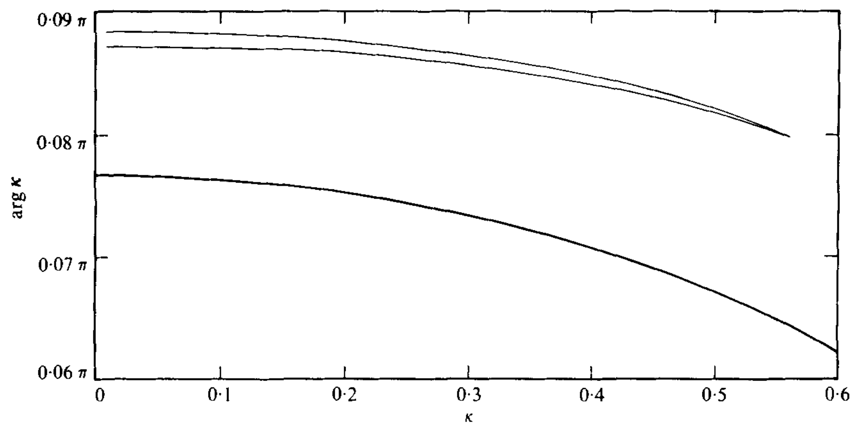

FIGURE 2. The region of instability for the permanent wave with $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}25$. The bold curve is the curve of linear resonance.

The bold curve is the curve of linear resonance [equation (1.1)] for the given value of μ. The points on the fine curves were calculated by the method outlined in § 2, further harmonics being added to greater precision until no further change occurred in the calculated values of λ_i. It was found that, at the smallest values of κ, eight harmonics calculated to a precision of 10^{-6}† were required to obtain values of λ_i that did not change when the precision was increased further. The analytical calculations cited in § 1 are valid only for the smallest values of κ, where they show that instability occurs in the neighbourhood of the bold curve in figure 1. A closer comparison is not possible because the analytical calculations assumed a Stokes wave with two harmonics, while the numerical calculations had to be continued to eight harmonics to obtain numerical convergence at small κ. The present calculations show that the region of instability extends to values of κ of about 0·5, i.e. to disturbances of twice the wavelength of the Stokes wave. The maximum rate of unstable growth is found to be $\lambda_i = 2{\cdot}65 \times 10^{-3}$, near $\kappa = 0{\cdot}3$. Figure 1 is consistent with the instability mechanism

† This refers to the absolute error in the Newton–Raphson calculation of the permanent wave harmonics.

described by Phillips (1974), the amplitude resonance causing a mismatch with the linear resonance condition.

The region of instability for the second permanent wave is sketched in figure 2. The fine curve is the outer bound of instability and the bold curve is the curve of linear resonance [equation (1.1)] for the given value of μ. The amplification rates are smaller than in the first example, the maximum amplification rate being $\lambda_i = 1{\cdot}2 \times 10^{-3}$, near $\kappa = 0{\cdot}35$. The previous analytical calculations predicting instability in the neighbourhood of the bold curve are not applicable here because, for the given wavelength, the permanent wave has too large an amplitude to be a Stokes wave.

Calculations were made also of part of the region of instability for a third permanent wave with $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}1$. This wave contains 11 harmonics exceeding 0·01. The region of instability was found to be further from the linear resonance curve, to be narrower and to have smaller rates of unstable amplification than in the previous examples. The smaller rates of amplification required the calculations to be extended to over 30 harmonics calculated to a precision of 10^{-6} in order to obtain numerical convergence.

4. Stability and instability

Now the time evolution of periodic disturbances of wavenumber 0·25 and amplitude $\hat{a}$ applied to the first two permanent waves of § 3 is calculated. The method of calculation is described in I (§ 4). The solutions are compared for the parallel stable case, when the margin of stability may be vanishingly small, and for the oblique unstable case, with the angle chosen as that at which the rate of amplification of the disturbance is greatest. The solutions are interpreted in the appendix.

The first permanent wave of § 3 is a Stokes wave, with $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}5$. When it is perturbed by the disturbance of wavenumber 0·25, i.e. of wavelength four times the fundamental wavelength, the perturbation harmonics about the fundamental grow according to

$$\hat{A}(0{\cdot}75, t) = \hat{a}[-1{\cdot}38 \exp(0{\cdot}155 i\epsilon t) + 1{\cdot}28 \exp(-0{\cdot}187 i\epsilon t) + 0{\cdot}10 \exp(-0{\cdot}589 i\epsilon t) + \ldots], \quad (4.1a)$$

$$\hat{A}(1{\cdot}25, t) = \hat{a}[1{\cdot}47 \exp(-0{\cdot}155 i\epsilon t) - 0{\cdot}63 \exp(0{\cdot}187 i\epsilon t) - 0{\cdot}84 \exp(0{\cdot}589 i\epsilon t) + \ldots]. \quad (4.1b)$$

[The magnitudes of the terms omitted in these and subsequent equations may be inferred by putting $t = 0$ and comparing each $\hat{A}$ with its initial value of zero. More detail is given in the appendix.]

When the Stokes wave is perturbed by the same disturbance at an angle $0{\cdot}136\pi$, the perturbation harmonics about the fundamental grow according to

$$\hat{A}(1 - 0{\cdot}25 \cos 0{\cdot}136\pi,\ -0{\cdot}25 \sin 0{\cdot}136\pi, t) = \hat{a}[(0{\cdot}23 + 0{\cdot}17i) \exp\{(0{\cdot}003 - 0{\cdot}402i)\epsilon t\} + (0{\cdot}23 - 0{\cdot}17i) \exp\{(-0{\cdot}003 - 0{\cdot}402i)\epsilon t\} - 0{\cdot}47 \exp(0{\cdot}637 i\epsilon t) + \ldots], \quad (4.2a)$$

$$\hat{A}(1 + 0{\cdot}25 \cos 0{\cdot}136\pi,\ 0{\cdot}25 \sin 0{\cdot}136\pi, t) = \hat{a}[(-0{\cdot}38 - 0{\cdot}21i) \exp\{(0{\cdot}003 + 0{\cdot}402i)\epsilon t\} + (-0{\cdot}38 + 0{\cdot}21i) \exp\{(-0{\cdot}003 + 0{\cdot}402i)\epsilon t) + 0{\cdot}75 \exp(-0{\cdot}637 i\epsilon t) + \ldots]. \quad (4.2b)$$

These are the two perturbation harmonics which benefit directly from the linear resonance (1.1).

The amplitudes of the exponentially increasing oscillations in (4.2) are less than the amplitudes of the lowest normal modes in (4.1) for $\exp(0{\cdot}003\epsilon t) < 4$ approximately, i.e. for $\epsilon t < 500$ approximately. This time span exceeds that of most practical applications. The model is also of doubtful validity at the end of this time span because of the neglect of the $O(\epsilon^2)$ tertiary interactions in the derivation of (2.2). The conclusion is that the effect of the stable disturbances in the same direction as the Stokes wave is of greater practical importance than the effect of the unstable disturbances oblique to the Stokes wave.

When the same calculations are performed on the second permanent wave of § 3, with $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}25$, the dominance of the parallel disturbances becomes more pronounced. The equivalent perturbation harmonics are now

$$\hat{A}(0{\cdot}75,t) = \hat{a}[-12{\cdot}65\exp(0{\cdot}013i\epsilon t) + 12{\cdot}38\exp(-0{\cdot}014i\epsilon t) + 0{\cdot}29\exp(-0{\cdot}346i\epsilon t) + \ldots], \tag{4.3a}$$

$$\hat{A}(1{\cdot}25,t) = \hat{a}[16{\cdot}02\exp(-0{\cdot}013i\epsilon t) - 14{\cdot}98\exp(0{\cdot}014i\epsilon t) - 1{\cdot}04\exp(0{\cdot}346i\epsilon t) + \ldots]. \tag{4.3b}$$

If the same disturbance is applied obliquely, the instability is a maximum at an angle $0{\cdot}0866\pi$, with perturbation harmonics

$$\hat{A}(1-0{\cdot}25\cos 0{\cdot}0866\pi,\ -0{\cdot}25\sin 0{\cdot}0866\pi, t) = \hat{a}[(0{\cdot}42+0{\cdot}32i)\exp\{(0{\cdot}001-0{\cdot}229i)\epsilon t\} + (0{\cdot}42-0{\cdot}32i)\exp\{(0{\cdot}001-0{\cdot}229i)\epsilon t\} - 0{\cdot}81\exp(0{\cdot}294i\epsilon t) + \ldots], \tag{4.4a}$$

$$\hat{A}(1+0{\cdot}25\cos 0{\cdot}0866\pi, 0{\cdot}25\sin 0{\cdot}0866\pi, t) = \hat{a}[(-0{\cdot}72+0{\cdot}01i)\exp\{(0{\cdot}001+0{\cdot}229i)\epsilon t\} + (-0{\cdot}72-0{\cdot}01i)\exp\{(-0{\cdot}001+0{\cdot}229i)\epsilon t\} + 1{\cdot}45\exp(-0{\cdot}294i\epsilon t) + \ldots]. \tag{4.4b}$$

Over the initial time span $\epsilon t < 500$ of the previous example, the amplitudes (4.4) of the unstable oscillations in this example are less than 10% of the amplitudes of the lowest normal modes of the stable oscillations in (4.3). A small parallel stable disturbance therefore gains energy from the permanent wave much faster than an oblique unstable disturbance over the time of validity or time of application of the present model. Hence a stable parallel disturbance can cause a much greater modification to a long permanent wave, in practice, than can an unstable oblique disturbance.

When further examples were calculated, it was found that the dominance of the parallel disturbances over the oblique disturbances increases as the wavelength of the permanent wave increases. For any given permanent wave, the dominance is approximately equivalent for different values of the disturbance wavenumber, provided that the angle of maximum oblique instability is chosen in each example. The reason for the dominance appears to be that the parallel disturbances interact near resonance with all the lower harmonics of the permanent wave, while the oblique disturbances interact resonantly with only the first two harmonics of the permanent wave.

5. Explanation of the instability

Phillips' explanation of the instability of Stokes waves to oblique disturbances is based on the occurrence of resonant triads among disturbance harmonics and the second harmonic of the permanent wave [equation (1.1)]. This instability persists for long permanent waves of greater amplitude than Stokes waves, so an attempt is now made to generalize Phillips' explanation.

An alternative description of (1.1) is that it determines the angle at which the disturbance of wavenumber $\boldsymbol{\kappa}$ must be offset from the permanent wave of wavenumber $\mathbf{k}$ in order that the disturbance harmonics of wavenumbers $\mathbf{k}-\boldsymbol{\kappa}$ and $\mathbf{k}+\boldsymbol{\kappa}$ interact to generate a harmonic proportional to $\exp i\{2\mathbf{k}.\mathbf{x}-[\omega(\mathbf{k}-\boldsymbol{\kappa})+\omega(\mathbf{k}+\boldsymbol{\kappa})]\,t\}$ which remains exactly in step with the second harmonic of the permanent wave proportional to $\exp i\{2(\mathbf{k}.\mathbf{x}-\omega(\mathbf{k})\,t)\}$. A resonant transfer of energy then occurs, causing exponential growth of the amplitudes of the disturbance harmonics until further growth is restricted by the bound on the total energy.

This explanation must be rephrased when it is applied to the second and third examples of permanent waves in § 3, because the frequency of the second harmonic of these permanent waves now differs significantly from the linear frequency $2\omega(\mathbf{k})$, and the horizontal velocity field caused by the permanent waves at the free surface contributes significantly to the frequencies of the disturbance harmonics. For the third permanent wave, for which $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}1$, the frequency $2\mathbf{i}_1.\mathbf{c}$ of the second harmonic is 2·063 compared with the linear frequency $2\omega(\mathbf{i}_1)$ of 1·997. The horizontal velocity field ranges from a maximum of 0·084 at the crests of the permanent wave to a minimum of $-0{\cdot}012$ on the horizontal intervals separating consecutive crests.

The explanation now is that $\boldsymbol{\kappa}$ can be offset from $\mathbf{k}$ at an angle such that the disturbance harmonic proportional to $\exp i\{2\mathbf{k}.\mathbf{x}-[\omega(\mathbf{k}-\boldsymbol{\kappa})+\omega(\mathbf{k}+\boldsymbol{\kappa})]\,t\}$, with the assistance of the horizontal velocity field at the free surface, remains exactly in step with the second harmonic of the permanent wave proportional to $\exp i\{2\mathbf{k}.(\mathbf{x}-\mathbf{c}t)\}$. The horizontal velocity field assists the positioning of the disturbance harmonic as follows. The region in front of a crest of the permanent wave is a region of velocity convergence, therefore shortening the horizontal length scale of any disturbance on the surface. If the disturbance is nearly stationary relative to the crest, then from the dispersion relation the shortening slows it and moves it back behind the crest. On the other hand, the region behind a crest of the permanent wave is a region of velocity divergence, therefore increasing the horizontal length scale of any disturbance on the surface. A disturbance nearly stationary relative to the crest increases its speed, from the dispersion relation, and moves to the front of the crest. The fine tuning which is possible by choice of the angle between $\boldsymbol{\kappa}$ and $\mathbf{k}$ enables the above disturbance harmonic, assisted by the horizontal velocity field, to remain nearly stationary relative to the permanent wave and to grow resonantly from it.

The author wishes to thank the referees for their comments, which have led to improvements in the presentation.

Appendix

The perturbation solutions (4.1)–(4.4) are interpreted here in terms of sum and difference interactions between the perturbation harmonics and the permanent wave harmonics. This interpretation should have been applied to the examples discussed in I (§ 6).

Equations (4.1) refer to a disturbance of wavenumber $\kappa = 0{\cdot}25$ applied in the same direction as the permanent wave. For this example, (2.5) (or equation (4.2) of I) is rewritten in scalar form as

$$\begin{aligned}\eta(x,t) &= \frac{1}{2}\sum_{k=1}^{\infty}\hat{B}(k-\kappa,t)\exp\{i(k-\kappa)x\}\\ &+\frac{1}{2}\sum_{k=0}^{\infty}a_k\exp\{ik(x-ct)\}\\ &+\frac{1}{2}\sum_{k=0}^{\infty}\hat{B}(k+\kappa,t)\exp\{i(k+\kappa)x\}+*.\end{aligned} \tag{A 1}$$

Equations (4.1) then become

$$\begin{aligned}\hat{B}(\kappa,t) = \hat{a}[&1{\cdot}20\exp(-0{\cdot}002i\epsilon t)\exp\{-i\omega(\kappa)t\}\\ &-0{\cdot}37\exp(0{\cdot}027i\epsilon t)\exp\{-i[c-\omega(1-\kappa)]t\}\\ &+0{\cdot}17\exp(-0{\cdot}034i\epsilon t)\exp\{-i[\omega(1+\kappa)-c]t\}+\ldots(7\times10^{-5})]\end{aligned}$$

(the bracketed number at the end of this and subsequent equations is the magnitude of the next term),

$$\begin{aligned}\hat{B}(1-\kappa,t) = \hat{a}[&-1{\cdot}38\exp(0{\cdot}002i\epsilon t)\exp\{-i[c-\omega(\kappa)]t\}\\ &+1{\cdot}28\exp(-0{\cdot}027i\epsilon t)\exp\{-i\omega(1-\kappa)t\}\\ &+0{\cdot}10\exp(0{\cdot}034i\epsilon t)\exp\{-i[2c-\omega(1+\kappa)]t\}+\ldots(4\times10^{-3})],\end{aligned}$$

$$\begin{aligned}\hat{B}(1+\kappa,t) = \hat{a}[&1{\cdot}47\exp(-0{\cdot}002i\epsilon t)\exp\{-i[c+\omega(\kappa)]t\}\\ &-0{\cdot}63\exp(0{\cdot}027i\epsilon t)\exp\{-i[2c-\omega(1-\kappa)]t\}\\ &-0{\cdot}84\exp(-0{\cdot}034i\epsilon t)\exp\{-i\omega(1+\kappa)t\}+\ldots(5\times10^{-4})],\end{aligned}$$

$$\begin{aligned}\hat{B}(2-\kappa,t) = \hat{a}[&-0{\cdot}62\exp(0{\cdot}002i\epsilon t)\exp\{-i[2c-\omega(\kappa)]t\}\\ &+0{\cdot}56\exp(-0{\cdot}027i\epsilon t)\exp\{-i[c+\omega(1-\kappa)]t\}\\ &+0{\cdot}04\exp(0{\cdot}034i\epsilon t)\exp\{-i[3c-\omega(1+\kappa)]t\}\\ &+0{\cdot}02\exp(-0{\cdot}022i\epsilon t)\exp\{-i\omega(2-\kappa)t\}+\ldots(9\times10^{-5})],\end{aligned}$$

$$\begin{aligned}\hat{B}(2+\kappa,t) = \hat{a}[&0{\cdot}41\exp(-0{\cdot}002i\epsilon t)\exp\{-i[2c+\omega(\kappa)]t\}\\ &-0{\cdot}18\exp(0{\cdot}027i\epsilon t)\exp\{-i[3c-\omega(1-\kappa)]t\}\\ &-0{\cdot}24\exp(-0{\cdot}034i\epsilon t)\exp\{-i[c+\omega(1+\kappa)]t\}\\ &+0{\cdot}003\exp(-0{\cdot}035i\epsilon t)\exp\{-i\omega(2+\kappa)t\}+\ldots(5\times10^{-5})].\end{aligned}$$

This solution may be interpreted as consisting of modes which approximate either the perturbation harmonics of wavenumber κ, $1-\kappa$ or $1+\kappa$ or modes which result from sum and difference interactions of these harmonics with the lower harmonics of the permanent wave. It is in a form which may be compared directly with the form assumed by Benjamin (1967, pp. 66–68) in his analysis of the stability of Stokes waves. The comparison is not good, because Benjamin used an ordering in which $\hat{B}(1 \pm \kappa)$ are $O(1)$ while $\hat{B}(\kappa)$ and $\hat{B}(2 \pm \kappa)$ are $O(\epsilon)$. Benjamin's initial condition is equivalent to $\hat{B}(1 \pm \kappa) = 1$, $\hat{B}(k \pm \kappa) = 0$ otherwise, but using this initial condition, $\hat{B}(\kappa)$ and $\hat{B}(1 \pm \kappa)$ were again found to be of the same magnitude at subsequent times. However, Benjamin also took κ to be $O(\epsilon)$, and when the calculations were repeated with $\boldsymbol{\kappa} = \epsilon = 0{\cdot}05$, the ordering did approximate that used by Benjamin.

Equations (4.2) refer to a disturbance of wavenumber $\boldsymbol{\kappa}$ of magnitude 0·25 applied at an angle $0{\cdot}136\pi$ to the direction of propagation of the permanent wave. For this example, (2.5) is rewritten as

$$\begin{aligned}\eta(\mathbf{x},t) &= \frac{1}{2}\sum_{k=1}^{\infty} \hat{B}(k\mathbf{i}_1-\boldsymbol{\kappa},t)\exp\{i(k\mathbf{i}_1-\boldsymbol{\kappa}).\mathbf{x}\}\\ &\quad+\frac{1}{2}\sum_{k=1}^{\infty} a_k\exp\{ik(x_1-ct)\}\\ &\quad+\frac{1}{2}\sum_{k=0}^{\infty} \hat{B}(k\mathbf{i}_1+\boldsymbol{\kappa},t)\exp\{i(k\mathbf{i}_1+\boldsymbol{\kappa}).\mathbf{x}\}+*. \end{aligned} \tag{A 2}$$

Equations (4.2) may then be expanded and rewritten as

$$\begin{aligned}\hat{B}(\boldsymbol{\kappa},t) &= \hat{a}\{0{\cdot}95\exp(-0{\cdot}050i\epsilon t)\exp\{-i\omega(\boldsymbol{\kappa})t\}\\ &\quad+[0{\cdot}05\cosh(0{\cdot}003\epsilon t)+0{\cdot}13i\sinh(0{\cdot}003\epsilon t)]\\ &\quad\times[p\exp(0{\cdot}129i\epsilon t)\exp\{-i[c-\omega(\mathbf{i}_1-\boldsymbol{\kappa})]t\}\\ &\quad+q\exp(-0{\cdot}090i\epsilon t)\exp\{-i[\omega(\mathbf{i}_1+\boldsymbol{\kappa})-c]t\}]+\ldots(5\times10^{-5})\}\end{aligned}$$

(the division of the unstable mode between the modes generated by the wavenumbers $\mathbf{i}_1 \pm \boldsymbol{\kappa}$ may be assigned arbitrarily, and is expressed here and in subsequent equations by the ratio $p:q$, where $p+q = 1$),

$$\begin{aligned}\hat{B}(\mathbf{i}_1-\boldsymbol{\kappa},t) &= \hat{a}\{-0{\cdot}47\exp(0{\cdot}050i\epsilon t)\exp\{-i[c-\omega(\boldsymbol{\kappa})]t\}\\ &\quad+[0{\cdot}47\cosh(0{\cdot}003\epsilon t)+0{\cdot}35i\sinh(0{\cdot}003\epsilon t)]\\ &\quad\times[p\exp(-0{\cdot}129i\epsilon t)\exp\{-i\omega(\mathbf{i}_1-\boldsymbol{\kappa})t\}\\ &\quad+q\exp(0{\cdot}090i\epsilon t)\exp\{-i[2c-\omega(\mathbf{i}_1+\boldsymbol{\kappa})]t\}]+\ldots(2\times10^{-6})\},\\ \hat{B}(\mathbf{i}_1+\boldsymbol{\kappa},t) &= \hat{a}\{0{\cdot}75\exp(-0{\cdot}050i\epsilon t)\exp\{-i[c+\omega(\boldsymbol{\kappa})]t\}\\ &\quad-[0{\cdot}75\cosh(0{\cdot}003\epsilon t)+0{\cdot}42i\sinh(0{\cdot}003\epsilon t)]\\ &\quad\times[p\exp(0{\cdot}129i\epsilon t)\exp\{-i[2c-\omega(\mathbf{i}_1-\boldsymbol{\kappa})]t\}\\ &\quad+q\exp(-0{\cdot}090i\epsilon t)\exp\{-i\omega(\mathbf{i}_1+\boldsymbol{\kappa})t\}]+\ldots(5\times10^{-4})\},\\ \hat{B}(2\mathbf{i}_1-\boldsymbol{\kappa},t) &= \hat{a}\{-0{\cdot}21\exp(0{\cdot}050i\epsilon t)\exp\{-i[2c-\omega(\boldsymbol{\kappa})]t\}\\ &\quad+[0{\cdot}19\cosh(0{\cdot}003\epsilon t)+0{\cdot}14i\sinh(0{\cdot}003\epsilon t)]\\ &\quad\times[p\exp(-0{\cdot}129i\epsilon t)\exp\{-i[c+\omega(\mathbf{i}_1-\boldsymbol{\kappa})]t\}\\ &\quad+q\exp(0{\cdot}090i\epsilon t)\exp\{-i[3c-\omega(\mathbf{i}_1+\boldsymbol{\kappa})t\}]\\ &\quad+0{\cdot}02\exp(-0{\cdot}028i\epsilon t)\exp\ -i\omega(2\mathbf{i}_1-\boldsymbol{\kappa})t+\ldots(8\times10^{-5})\},\\ \hat{B}(2\mathbf{i}_1+\boldsymbol{\kappa},t) &= \hat{a}\{0{\cdot}21\exp(-0{\cdot}050i\epsilon t)\exp\{-i[2c+\omega(\boldsymbol{\kappa})]t\}\\ &\quad-[0{\cdot}21\cosh(0{\cdot}003\epsilon t)+0{\cdot}12i\sinh(0{\cdot}003\epsilon t)]\\ &\quad\times[p\exp(0{\cdot}129i\epsilon t)\exp\{-i[3c-\omega(\mathbf{i}_1-\boldsymbol{\kappa})]t\}\\ &\quad+q\exp(-0{\cdot}090i\epsilon t)\exp\{-i[c+\omega(\mathbf{i}_1+\boldsymbol{\kappa})]t\}]\\ &\quad+0{\cdot}003\exp(-0{\cdot}036i\epsilon t)\exp\{-i\omega(2\mathbf{i}_1+\boldsymbol{\kappa})t\}+\ldots(5\times10^{-5})\}.\end{aligned}$$

The structure of the solution in this oblique case is similar to that for the parallel case except for the unstable mode. The unstable mode is associated with the modes generated by the wavenumbers $\mathbf{i}_1 \pm \boldsymbol{\kappa}$, as expected for the reasons advanced in § 1.

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